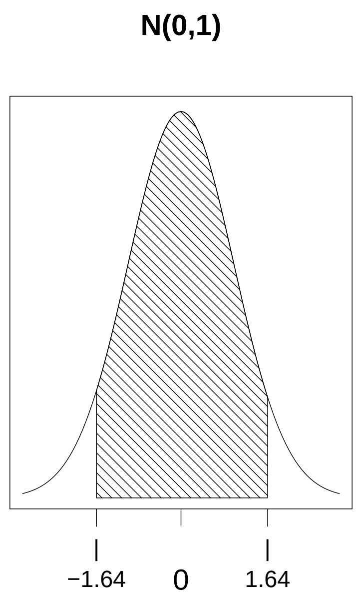
Confidence Intervals for STAT112

Charles Fleming

January 20, 2015

Problem 1. *In a laboratory, a sample of 30 specimens were weighed. The specimens came from a population with mean=10 grams with standard deviation of 1 gram. Let $\bar{x}$ denote the sample mean of the 30 weights. Find $P(9.7 \leq \bar{x} \leq 10.3)$.*



The distribution of a measurement is unknown. Although, according the Central Limit Theorem, the sample z-score will converge to N(0,1) only when the number of observations goes to infinity, we will, nonetheless, rely on the Central Limit Theorem even when the number of observations is only 30 and use the Normal distribution for the sample z-score. In fact, we will go further and assume that, $X_i \sim N(10, 1^2)$. Therefore, $E[\bar{x}] = \mu = 10$ and $var(\bar{x}) = \frac{\sigma^2}{n} = \frac{1}{30}$; hence $\bar{x} \sim N(10, \frac{1}{30})$ and we will proceed to answer the question in terms of z-scores.

$$
\begin{aligned}
P(9.7 \leq \bar{x} \leq 10.3) &= P\left(\frac{9.7-10}{\sqrt{\frac{1}{30}}}) \leq \frac{\bar{x}-10}{\sqrt{\frac{1}{30}}} \leq \frac{10.3-10}{\sqrt{\frac{1}{30}}}\right) \\
&= P\left(\frac{-.3}{.1825} \leq z \leq \frac{.3}{.1825}\right) \\
&= P(-1.64 \leq z \leq 1.64) \\
&= 1 - 2P(z \geq 1.64) = 1 - 2(.051) \\
&= .898
\end{aligned}
$$

The answer to the problem is that the probability that the average weight of any sample of 30 specimens from that laboratory will lie within the interval, (9.7, 10.3), is .898. On the other hand, the probability that the weight of a particular specimen will lie in the interval, (9.7, 10.3), is not the same .898. That can be easily demonstrated by the following equivalences. As before, it is assumed that $X_i \sim N(10, 1^2)$, this time n=1.

$$
\begin{aligned}
P(9.7 \leq x_i \leq 10.3) &= P\left(\frac{9.7-10}{\sqrt{1}}) \leq \frac{\bar{x}-10}{\sqrt{1}} \leq \frac{10.3-10}{\sqrt{1}}\right) \\
&= P\left(-.3 \leq z \leq .3\right) \\
&= 2(.11791) \\
&= .236
\end{aligned}
$$

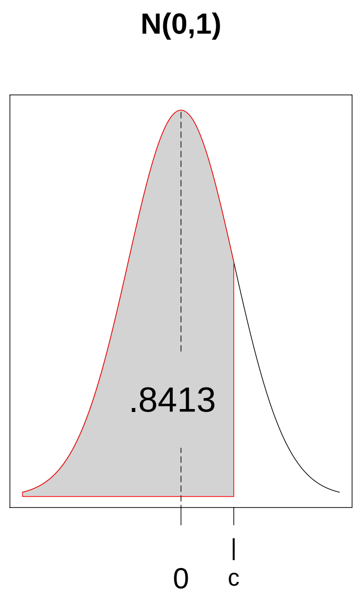
The observation that the average weight of numerous specimens and the weight of a particular specimen following different distributions raises a natural question: What would be the right interval in which the weight of a particular specimen will lie with a probability of .898? Consider the interval, (8.36, 11.64). Then

$$
\begin{aligned}
P(8.36 \leq x_i \leq 11.64) &= P\left(\frac{8.36-10}{\sqrt{1}}) \leq \frac{\bar{x}-10}{\sqrt{1}} \leq \frac{11.64-10}{\sqrt{1}}\right) \\
&= P\left(-1.64 \leq z \leq 1.64\right) \\
&= 2(.44950) \\
&= .899
\end{aligned}
$$

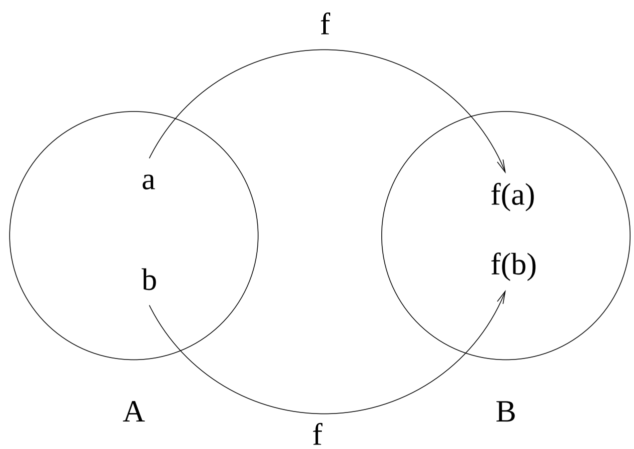
The probability that $\bar{x}$ is in (9.7, 10.3) is .898, and the probability that x_i is in (8.36, 11.64) is .899. The precision of an estimate like the average weight is indicated by the width of an interval which in a certain sense brackets the estimate with a specified probability. That such an interval can be used to gauge the precision of an estimate provides a way to compare one estimate against another. The estimate with the shortest interval is deemed to be the best one; therefore, the sample average must be a more precise estimate than a particular measurement and that agrees with our intuition. The method of constructing these intervals for an estimate forms the subject of confidence intervals.

1 z Quantile

More often than not, finding the inverse of a probability is required for doing common statistical problems than computing a probability. The inverse of a probability is that z which will produce the given probability.

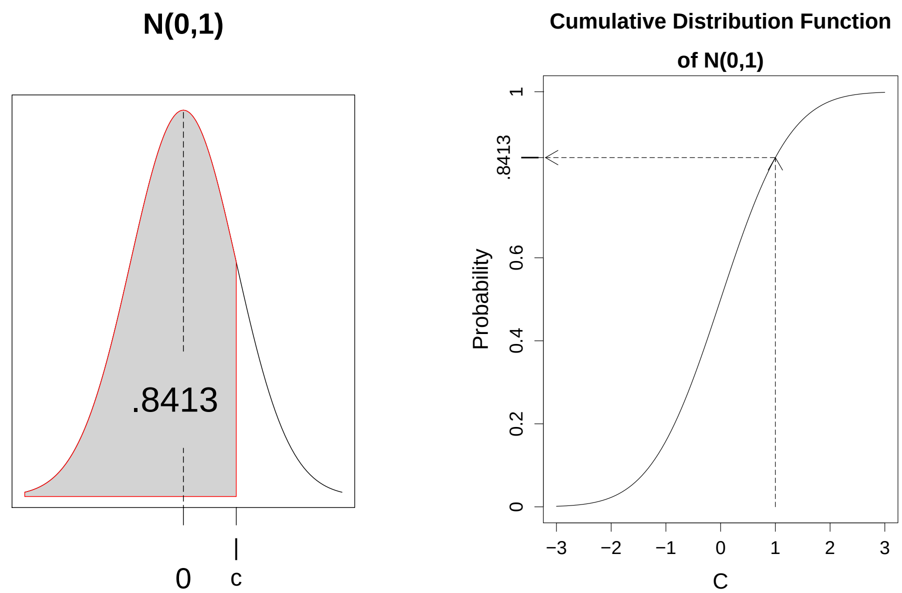


For example, let $z \sim N(0,1)$; it is desired to find c such that $P(z \leq c) = .84134$. Given the probability to be .84134, what number, c, produced it? The table of probabilities for the Standard Normal distribution found in Appendix 3 gives the area under the curve from 0 to c so that the remainder of .84134 lying to the right of 0 is .34134 which was found by having taken into account the .5 which lies to the left of 0. The number which comes closest to .34134 in the body of the table of probabilities is co-incidentally .34134, and it corresponds to z=1. The answer to the question is: c=1. A check of the answer verifies that $P(z \leq 1) = .8413$. This process of finding the inverse probability falls under the theory of inverse functions.

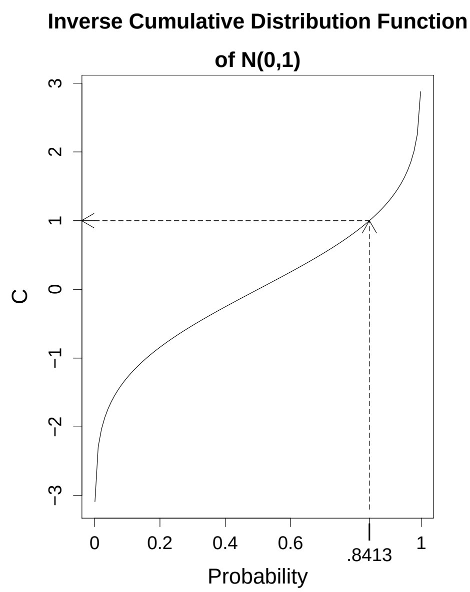


Definition 1. *A **function** is a mapping from set A into set B such that if $f(a) \neq f(b)$ then $a \neq b$.*

The definition is saying that, a function is not supposed to map one point to different numbers, although it is certainly permissible for a function to map two or more different points to the same image. Those functions which produce a one-to-one correspondence between two sets are special. If a function, f, is one-to-one then it has an inverse which is commonly denoted by f^{-1}. Because the cumulative probability function, $P(X \leq c)$, is a function which is one-to-one, it has an inverse. Since our interest will usually focus on the Normal distribution or a distribution that is based on it, let us study the inverse of the cumulative distribution of the Normal distribution. If $z \sim N(0,1)$, then, in terms of integral calculus, the area under the curve up to c is: $P(X \leq c) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{c} e^{-\frac{t^2}{2}} dt$. $P(X \leq c)$ is a function in c as can be seen in this equation. A picture of the cumulative distribution of N(0,1) is shown on the right while its probability density function with which we are intimately familiar is on the left.



The cumulative distribution function (CDF) expresses the area under the probability density function from $-\infty$ up to c. If c=1, then by using the probability table for N(0,1), $P(z \leq 1) = .8413$ which is the area under the probability density function from $-\infty$ to 1. By referring to the cumulative distribution function of N(0,1) shown above, when c=1, then the function gives the answer: .8413. Every point on the CDF as shown on the right corresponds to the area under the curve given on the left. The picture of the cumulative distribution function of N(0,1), the area under the probability density function shown on the left, and the table of probabilities of N(0,1) given in Appendix 3 are actually the same. The picture of the CDF makes it more obvious than the table that the cumulative distribution function is a one-to-one function; consequently, it has an inverse. A graph of the inverse function of the Normal CDF is shown below:



To find a number, c, such that $P(z \leq c) = .8413$, the graph of the inverse cumulative distribution function shows that c=1. The graph of the inverse cumulative distribution function is the same as the cumulative distribution function but rotated 90^o and flipped over along the x axis. Consequently, the graph of the inverse function does not provide any more information than the graph of the cumulative distribution function itself. Even though the concept of an inverse function is essential in deriving confidence intervals, its picture is ignored in practice. For the purpose of finding probabilities and for finding the inverse of the probabilities, the table of probabilities for N(0,1) is sufficient.

To find that c which produces a specified probability, we search in the body of the table for the probability. However, the table is constructed to provide the area between 0 and c. In order to utilize the table, it is necessary then to search for the remainder of the area lying to the right of 0. The area to the left of 0 is .5, consequently, we subtract .5 from the specified probability to give the remainder of the area to the right of 0 and that is the value we seek in the body of the table.

Example 1. 1. *Find c such that* $P(z \leq c) = .5 = .5 + 0 \rightarrow c = 0$.

2. *Find c such that* $P(z \leq c) = .69146 = .5 + .19146 \rightarrow c = .5$

3. *Find c such that* $P(X \leq c) = .9750 = .5 + .4750 \rightarrow c = 1.96$. *Check* $P(z \leq 1.96) = .5 + .4750 = .9750$.

It is tedious to keep writing: *Find c such that* $P(X < c) = .9750$, for instance. A shorthand notation has been developed to use in its place.

Definition 2. *If* $z \sim N(0,1)$, *denote by* $\boldsymbol{z_\alpha}$ *that number such that* $P(X \leq z_\alpha) = 1 - \alpha$.

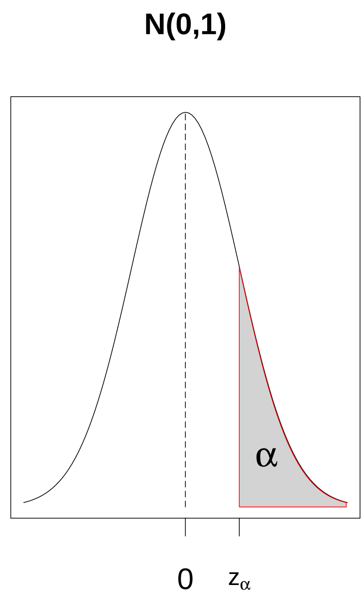


Figure 1

Example 2. *Let* $\alpha = .08$*; find* $z_{.08}$*. By definition,* $P(X \leq z_{.08}) = 1 - .08 = .92 = .5 + .42 \rightarrow z_{.08} = 1.41$*. A check of the answer,* $P(z \leq 1.41) = .92$*, verifies that it is correct.*

Simple mathematical notation like z_α succinctly facilitates the incorporation of the inverse of a cumulative distribution into statistical formulas. To emphasis the utility of the notation, z_α, the previous three examples will be done again in terms of z_α.

1. $P(z \leq c) = .5 = 1 - .5 = P(z \leq z_{.5}) \rightarrow c = z_{.5} = 0$.

2. $P(z \leq c) = .6915 = 1 - .3085 = P(z \leq z_{.3085}) \rightarrow c = z_{.3085} = .5$.

3. $P(z \leq c) = .9750 = 1 - .025 = P(z \leq z_{.025}) \rightarrow c = z_{.025} = 1.96$.

An alternative and usually more convenient definition of z_α is given by the following theorem.

Theorem 1. $P(z \geq z_\alpha) = \alpha$.

Proof. $P(z \geq z_\alpha) = 1 - P(z \leq z_\alpha) = 1 - (1 - \alpha) = \alpha$. ∎

One can think of z_α as that number such that the area under the curve to the right of it is equal to α and that idea is shown schematically in Figure 1.

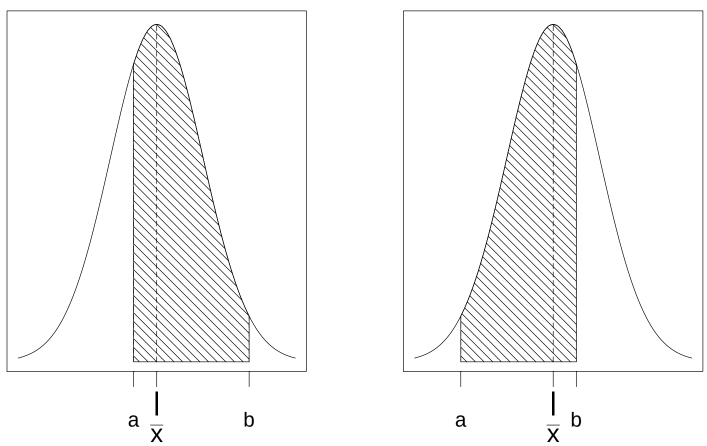
Example 3. *Find* $z_{.025}$*.* $P(z \geq z_{.025}) = .025 \rightarrow P(z \leq z_{.025}) = .975 = .5 + .475 \rightarrow z_{.025} = 1.96$*. A check of the answer,* $P(z \geq 1.96) = .025$*, verifies answer.*

Definition 3. $\boldsymbol{z_\alpha}$ *is called a quantile of the Standard Normal distribution.*

The sample mean, $\bar{x} = \frac{\sum_{i \in \mathcal{S}} x_i}{n}$, is a random variable. As such, it changes depending on the luck of drawing the sample. From the Central Limit Theorem, the graph which approximates the distribution of $\bar{x}$ indicates that $\bar{x}$ spends most of its time in and around the central peak: that is, near its expected value. A question worth pondering concerns the idea of finding two numbers about the central peak such that the resulting interval will capture, so to speak, $\bar{x}$ with a certain probability of $1 - \alpha$.

Problem 2. *Find two numbers, a and b, such that* $P(a \leq \bar{x} \leq b) = 1 - \alpha$.

Unfortunately, there is no unique answer to this problem because there are infinite number of possible a's and b's which will do the job as the following two graphs suggest. In both, the areas under the curves are the same but their end points are obviously different.



It is necessary, therefore, to impose another constraint in order to obtain a unique lower limit and a unique upper limit of the interval. This is done by taking advantage of the symmetry of the Normal distribution by choosing a and b to be symmetric about $\bar{x}$ such that $P(a \leq \bar{x} \leq b) = 1 - \alpha$. The consequence of requiring a and b to be placed symmetrically about $\bar{x}$ along with the symmetry of the Normal distribution assures us by means of differential calculus that the length of the resulting interval will be a minimum. It will be the shortest interval which will bracket $\bar{x}$ with the specified probability, $1 - \alpha$. The lower and upper limits of this interval is given by the next theorem.

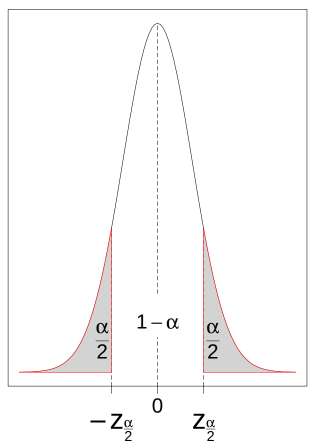
Theorem 2. *If* X_i *are i.i.d* $N(\mu, \sigma^2)$, $\bar{x} = \frac{X_1 + \cdots + X_n}{n}$, $P(a \leq \bar{x} \leq b) = 1 - \alpha$, *and a and b are symmetric about* $\bar{x}$, *then*

1. $a = \mu - \frac{\sigma}{\sqrt{n}} z_{\frac{\alpha}{2}}$

2. $b = \mu + \frac{\sigma}{\sqrt{n}} z_{\frac{\alpha}{2}}$

Proof. Let a and b be two numbers symmetrically placed about $\bar{x}$ such that $P(a \leq \bar{x} \leq b) = 1 - \alpha$. Subtracting the expected value of $\bar{x}$ from both sides and dividing by the standard deviation of $\bar{x}$ produces:

$$\begin{aligned} P\left(\frac{a-\mu}{\frac{\sigma}{\sqrt{n}}} \leq \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}} \leq \frac{b-\mu}{\frac{\sigma}{\sqrt{n}}}\right) &= 1-\alpha \\ P\left(\frac{a-\mu}{\frac{\sigma}{\sqrt{n}}} \leq z \leq \frac{b-\mu}{\frac{\sigma}{\sqrt{n}}}\right) &= 1-\alpha \end{aligned}$$



Symmetric about $\bar{x}$ means that $\left(\frac{a-\mu}{\frac{\sigma}{\sqrt{n}}}\right) = -\left(\frac{b-\mu}{\frac{\sigma}{\sqrt{n}}}\right)$; consequently,

$$\begin{aligned} P\left(-\left(\frac{b-\mu}{\frac{\sigma}{\sqrt{n}}}\right) \leq z \leq \frac{b-\mu}{\frac{\sigma}{\sqrt{n}}}\right) &= 1-\alpha \\ P\left(-z_{\frac{\alpha}{2}} \leq z \leq z_{\frac{\alpha}{2}}\right) &= 1-\alpha \rightarrow \frac{b-\mu}{\frac{\sigma}{\sqrt{n}}} = z_{\frac{\alpha}{2}} \rightarrow b = \mu + \frac{\sigma}{\sqrt{n}} z_{\frac{\alpha}{2}} \end{aligned}$$

and since $\left(\frac{a-\mu}{\frac{\sigma}{\sqrt{n}}}\right) = -z_{\frac{\alpha}{2}} \rightarrow a = \mu - \frac{\sigma}{\sqrt{n}} z_{\frac{\alpha}{2}}$

Therefore, $P(\mu - \frac{\sigma}{\sqrt{n}} z_{\frac{\alpha}{2}} \leq \bar{x} \leq \mu + \frac{\sigma}{\sqrt{n}} z_{\frac{\alpha}{2}}) = 1 - \alpha$. ■

Example 4. *Suppose a sample consists of 25 elements where $X_i \sim N(10, 9)$.*

1. *Find two numbers, a and b, which are symmetric about $\bar{x}$ such that the area under the curve lying between them is 95%.*

(a) *95%=100(1-.05)%* $\rightarrow \alpha = .05$, *hence* $\frac{\alpha}{2} = .025$.

(b) $z_{\frac{\alpha}{2}} = z_{.025} = 1.96$.

(c) $a = \mu - \frac{\sigma}{\sqrt{n}} z_{\frac{\alpha}{2}} = 10 - \frac{3}{5}(1.96) = 8.825$
$b = \mu + \frac{\sigma}{\sqrt{n}} z_{\frac{\alpha}{2}} = 10 + \frac{3}{5}(1.96) = 11.176$

The meaning of these two numbers must be viewed in the context of performing many replicated experiments. If the same experiment is performed 100 times, and they are done independently of each other, then associated with each sample, i, there is an $\bar{x}_i$. *On the average, 95 of the* $\bar{x}_i$*'s will lie between 8.825 and 11.176. Even before any experimentation is conducted, the anticipation that 95 percent of the* $\bar{x}_i$*'s will lie between 8.825 and 11.176 is valid. The ability to predict with a certain level of confidence that random variables will likely be found within an interval of prescribed length makes these two numbers, a and b, very useful.*

2. *Do the same experiment except that the area under the curve lying between a and b is 90%.*

(a) $90\% = 100(1 - .10)\% \rightarrow \alpha = .10$, *hence* $\frac{\alpha}{2} = .05$.

(b) $z_{.05} = 1.64$.

(c) $a = \mu - \frac{\sigma}{\sqrt{n}} z_{\frac{\alpha}{2}} = 10 - \frac{3}{5}(1.64) = 9.016$
$b = \mu + \frac{\sigma}{\sqrt{n}} z_{\frac{\alpha}{2}} = 10 + \frac{3}{5}(1.64) = 10.984$

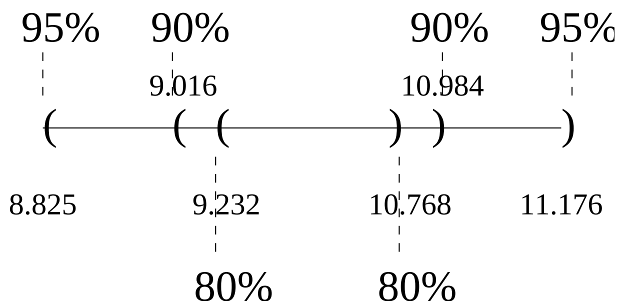
3. *Again, but with an area of 80%.*

(a) $80\% = 100(1 - .20)\% \rightarrow \alpha = .20$, *hence* $\frac{\alpha}{2} = .10$.

(b) $z_{.10} = 1.28$.

(c) $a = \mu - \frac{\sigma}{\sqrt{n}} z_{\frac{\alpha}{2}} = 10 - \frac{3}{5}(1.28) = 9.232$
$b = \mu + \frac{\sigma}{\sqrt{n}} z_{\frac{\alpha}{2}} = 10 + \frac{3}{5}(1.28) = 10.768$

Each section of the preceding example only employs a different α. As α changes from .05 to .20, the distance between a and b decreases. It stands to reason that the more stringent the requirement of capturing $\bar{x}$ within brackets, the wider the spacing of the brackets must become, as the following diagram shows.



2 Confidence Interval when σ^2 Is Known

Suppose that the variance is known but μ is unknown, specifically $X_i \sim N(\mu, 25)$. In this case, an estimate of μ can, nevertheless, be obtained from the experimental data but at the sacrifice of giving up some precision in the estimate as we shall soon see. From our discussions of estimation and of sampling distributions which culminated in Theorem 3[1], we can assert that $\hat{\mu} = \bar{x}$ and $var(\bar{x}) = \frac{\sigma^2}{n}$. Fortunately, in this present case, σ^2 and n are given, even though μ is unknown.

It was already shown in Theorem 2 that $P(\mu - \frac{\sigma}{\sqrt{n}} z_{\frac{\alpha}{2}} \leq \bar{x} \leq \mu + \frac{\sigma}{\sqrt{n}} z_{\frac{\alpha}{2}}) = 1 - \alpha$. By using simple algebraic manipulations of subtracting $\bar{x}$ and μ from all sides and multiplying all sides by -1, this expression can be rearranged so that:

$$P(-\bar{x} - \frac{\sigma}{\sqrt{n}} z_{\frac{\alpha}{2}} \leq -\mu \leq -\bar{x} + \frac{\sigma}{\sqrt{n}} z_{\frac{\alpha}{2}}) = P(\bar{x} - \frac{\sigma}{\sqrt{n}} z_{\frac{\alpha}{2}} \leq \mu \leq \bar{x} + \frac{\sigma}{\sqrt{n}} z_{\frac{\alpha}{2}}) = 1 - \alpha$$

The two numbers which straddle μ define the lower and upper limits of a confidence interval.

Definition 4. *The interval* $(\bar{x} - \frac{\sigma}{\sqrt{n}} z_{\frac{\alpha}{2}}, \bar{x} + \frac{\sigma}{\sqrt{n}} z_{\frac{\alpha}{2}})$ *is called the* ***100(1-α)% confidence interval*** *of μ. Confidence interval is commonly abbreviated by CI.*

The concept of a confidence interval was published in 1934 by Jerzy Neyman. He was also responsible for creating a rigorous method of testing a statistical hypothesis which will constitute the next chapter. The importance of the concept of confidence intervals cannot be overemphasized. It is the most important concept in this course, and it lies at the foundation of making inferences about a population from a sample.

---

[1]

Theorem 3. *If X_1, X_2, ..., X_n are i.i.d. each with mean μ and variance σ^2, and $\bar{x} = \frac{X_1 + \cdots + X_n}{n}$, then*

$$E[\bar{x}] = \mu \text{ and } var(\bar{x}) = \frac{\sigma^2}{n}$$

Example 5. *If it is assumed that $x_i \sim N(\mu, 25)$ during the performance of an experiment in which 100 samples are drawn from a population with population mean μ and population variance 25, then the construction of a 95% confidence interval follows a simple recipe. Suppose the sample mean is: $\bar{x}$ =85.*

1. *95%=100(1-α)% $\rightarrow \alpha = .05$, hence $\frac{\alpha}{2} = .025$.*
2. $z_{.025} = 1.96$.
3. *lower limit,* a=$\bar{x} - \frac{\sigma}{\sqrt{n}} z_{\frac{\alpha}{2}} = 85 - \frac{\sqrt{25}}{\sqrt{100}} 1.96 = 84.02$

   upper limit, b=$\bar{x} + \frac{\sigma}{\sqrt{n}} z_{\frac{\alpha}{2}} = 85 + \frac{\sqrt{25}}{\sqrt{100}} 1.96 = 85.98$
4. *95% CI of μ =(84.02,85.98).*

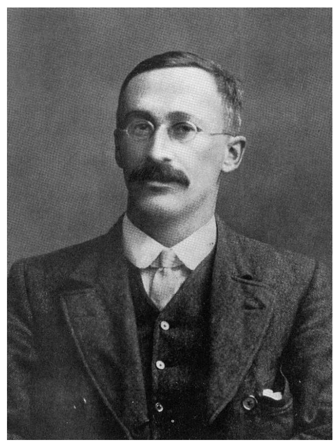
What does this mean? If 100 experiments are performed, 95 of the resulting confidence intervals will straddle the population mean. This is a profound statement to make because it implies that, about the population mean, that unknown quantity which is the cause of doing the experiment in the first place, an interval can be placed with a certain prescribed probability, $1 - \alpha$. It is impossible to tell if the confidence interval does indeed cover the population mean, yet, if 100 independent and identical experiments were to be performed, on the average, 95 confidence intervals will cover the population mean. A confidence interval is the best bet of a pragmatist for locating the mean of the population.

3 Confidence Interval when σ^2 Is Unknown

In practice, both the population mean and the population variance are unknown otherwise there is no purpose for conducting an experiment. An experiment is designed and conducted for the sole purpose of estimating a certain attribute of the population. The most commonly sought after characteristics of a population are the population mean and the population variance. Even more disconcerting than the lack of knowledge of μ and σ^2, is not knowing how X_i is distributed which in turn makes the distributions of the sample mean and the sample variance unknown. Nevertheless, if the sampling is done randomly and independence of taking measurements is guaranteed, then the conditions are met to invoke the Central Limit Theorem. Then by means of the Central Limit Theorem, the Standard Normal distribution can be used to approximate the distribution of the sample z-score. As more observations are made, the better the sample z-score, $\frac{\bar{x}-\mu}{\frac{s}{\sqrt{n}}}$ approximates N(0,1).

Before addressing the general case in which the distribution of X is unknown, the simpler one in which it is asserted that $X_i \sim N(\mu, \sigma^2)$ will be discussed. In this case, μ and σ^2 are not known but we do know that $X_i \sim N(\mu, \sigma^2)$. Because μ and σ^2 are unknown, we are forced to estimate: μ by $\bar{x}$ and σ^2 by s^2. In accordance with the method of constructing confidence intervals, the same procedure will be used as before; that is, two numbers, a and b, will be computed such that $P(a \leq \bar{x} \leq b) = 1 - \alpha$. Mathematically, the two numbers must be symmetrically placed about $\bar{x}$ such that

$$\begin{aligned} P(a \leq \bar{x} \leq b) &= 1 - \alpha \\ P\left(\frac{a-\mu}{\frac{s}{\sqrt{n}}} \leq \frac{\bar{x}-\mu}{\frac{s}{\sqrt{n}}} \leq \frac{b-\mu}{\frac{s}{\sqrt{n}}}\right) &= 1 - \alpha \end{aligned}$$



William Sealy Gosset
1876-1934

However, $\frac{\bar{x}-\mu}{\frac{s}{\sqrt{n}}} \nsim N(0,1)$, even though $\frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$. The difference between the two lies in the denominator. In the z-score, σ is a constant, while in the sample z-score, s is a random variable. Although $\frac{\bar{x}-\mu}{\frac{s}{\sqrt{n}}}$ is the sample z-score, and it converges to the Standard Normal via the Central Limit Theorem, it is not distributed as a N(0,1). Instead, it follows a different probability distribution. In the special case when $X_i \sim N(\mu, \sigma^2)$, it is not necessary to approximate the distribution of the sample z-score by the Central Limit Theorem because the assumption that $X_i \sim N(\mu, \sigma^2)$ permits the derivation of an exact distribution for $\frac{\bar{x}-\mu}{\frac{s}{\sqrt{n}}}$. It is called the Student's t distribution with n-1 degrees of freedom. This distribution was derived by William Sealy Gosset around 1908. He showed that $\frac{\bar{x}-\mu}{\frac{s}{\sqrt{n}}} \sim t_{n-1}$. The subscript, n-1, is called the degrees of freedom.

Recall that the formula for the sample variance is: $s^2 = \frac{\sum_{i \in S}(x_i - \bar{x})}{n-1}$. The n-1 appears in the denominator of s^2 to make it an unbiased estimator of σ^2; it is the same value for the degrees of freedom for the t distribution. A picture of two versions of the t distribution for 2 and 10 degrees of freedom is given respectively by the dotted and dashed curves in Figure

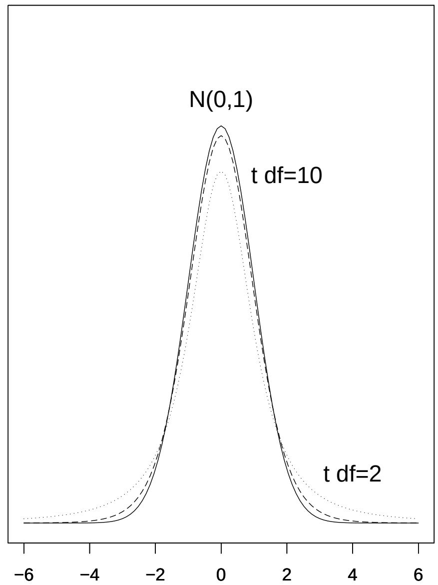


Figure 2

2. The N(0,1) is represented by a solid curve to which the t distribution converges as the degrees of freedom increase to infinity. That bell shaped feature causes the t distribution to bear a close resemblance to the Normal distribution; in fact, as the degrees of freedom approaches ∞, the t distribution will converge to the N(0,1) distribution. The t distribution is also symmetric about zero like the Standard Normal distribution, but the t distribution is flatter and its tails are heavier than the Normal distribution. The symmetry about zero is a useful feature for by imposing the same constraint on a and b to be symmetric about the mean, the derivation of a confidence interval becomes possible.

Theorem 4. *If X_i are i.i.d. $N(\mu, \sigma^2)$, then*

$$P(\bar{x} - \frac{s}{\sqrt{n}} t_{n-1,\frac{\alpha}{2}} \leq \mu \leq \bar{x} + \frac{s}{\sqrt{n}} t_{n-1,\frac{\alpha}{2}}) = 1 - \alpha.$$

Definition 5. *The interval $(\bar{x} - \frac{s}{\sqrt{n}} t_{n-1,\frac{\alpha}{2}}, \bar{x} + \frac{s}{\sqrt{n}} t_{n-1,\frac{\alpha}{2}})$ is called the* ***100(1-α)% confidence interval*** *of μ where the X_i's are i.i.d. Normal with mean μ and variance σ^2.*

Definition 6. $\boldsymbol{t_{n,\alpha}}$ *is called a quantile of the Student's t distribution, t_n.*

By being a quantile, it means that $P(t_n \leq t_{n,\alpha}) = 1 - \alpha$ or equivalently, $P(t_n > t_{n,\alpha}) = \alpha$.

The t distribution has a rather colorful history. It is called Student's t because the Guinness Brewery where Gosset worked as a statistician did not permit its employees to publish under their real names. Although the brewery, the same one which publishes the *Guinness Book of World Records*, encouraged its employees to publish in professional journals, the company was fearful that a reader of an article could infer the substance of some trade secret of the company. Instead, the Guinness Brewery encouraged the use of pseudonyms. Gosset chose the name Student. Only a select few people knew the true identity of Student. The identity of Student was not revealed until the 1930's when the restriction was finally lifted by Guinness, and it was safe to tell his real name.

Unlike the property of a Normal random variable which can be transformed into the Standard Normal distribution by means of the z-score, a random variable which follows a Student's t distribution cannot be so transformed to a standard distribution. There is no standard t distribution. For each degree of freedom there is a distinct t distribution. There must be a separate table of probabilities for each degree of freedom resulting in an unpleasant prospect of publishing a hundred page appendix in a statistics book only to accommodate the tables for the t distribution alone.

About the only time a t distribution is used is for getting quantiles for constructing confidence intervals and testing hypotheses. By convention, due to a feud between Karl Pearson and Ronald Fisher, two very prominent fathers of modern statistics, quantiles for α= .20, .10, .05, .025, .01, and .001 are widely used. With ready access to computers, this convention is giving way to the use of any quantile and the prospect of dispensing with tables all together might seem an imminent reality. Nonetheless, oftentimes it is more convenient to open a book, inspect a table, and go on with your work instead of anxiously waiting for a computer to warm-up, login, and to start a statistical software package just to get a quantile. The convention of using only a few α's, reduces a hundred tables to one table. This table which appears in Appendix 3 consists of six columns and about one hundred rows, one for each degree of freedom. Obtaining a t quantile is easy; for example, $t_{24,.025} = 2.064$. It is found by following row $\nu = 24$ until column $t_{.025}$ is reached.

Example 6. *Suppose a sample consists of 25 elements for which $X_i \sim N(\mu, \sigma^2)$. Furthermore, it was found that $\hat{\mu} = \bar{x} = 10$ and that $\widehat{\sigma^2} = s^2 = 9$.*

1. *Find 95% confidence interval for μ.*
   (a) *95%=100(1-α)% $\rightarrow \alpha = .05$, hence $\frac{\alpha}{2} = .025$ and n=25; therefore, n-1=24.*
   (b) $t_{n-1,\frac{\alpha}{2}} = t_{24,.025} = 2.064$.
   (c) *lower limit,* $a = \bar{x} - \frac{s}{\sqrt{n}} t_{n-1,\frac{\alpha}{2}} = 10 - \frac{3}{5}(2.064) = 8.7616$.
   upper limit, $b=\bar{x} + \frac{s}{\sqrt{n}} t_{n-1,\frac{\alpha}{2}} = 10 + \frac{3}{5}(2.064) = 11.2384$.
   (d) *95% CI of μ =(8.7616,11.2384).*
2. *Continuation of the previous example but need to find 90% CI.*

(a) *90%=100(1-α)* $\rightarrow \alpha = .10$, *hence* $\frac{\alpha}{2} = .05$ *and as before n-1=24.*

(b) $t_{n-1,\frac{\alpha}{2}} = t_{24,.05} = 1.711$.

(c) *lower limit,* $a = \bar{x} - \frac{s}{\sqrt{n}} t_{n-1,\frac{\alpha}{2}} = 10 - \frac{3}{5}(1.711) = 8.9734$.
upper limit, $b = \bar{x} + \frac{s}{\sqrt{n}} t_{n-1,\frac{\alpha}{2}} = 10 + \frac{3}{5}(1.711) = 11.0266$.

(d) *90% CI of* μ *=(8.9734,11.0266).*

3. *Final continuation of the previous example but need to find 80% CI.*

(a) *80%=100(1-α)* $\rightarrow \alpha = .20$ *hence* $\frac{\alpha}{2} = .10$ *and as before n-1=24.*

(b) $t_{n-1,\frac{\alpha}{2}} = t_{24,.10} = 1.318$.

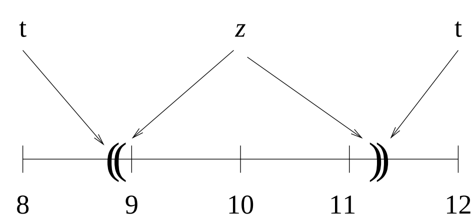
(c) *lower limit,* $a = \bar{x} - \frac{s}{\sqrt{n}} t_{n-1,\frac{\alpha}{2}} = 10 - \frac{3}{5}(1.318) = 9.2092$.
upper limit, $b = \bar{x} + \frac{s}{\sqrt{n}} t_{n-1,\frac{\alpha}{2}} = 10 + \frac{3}{5}(1.318) = 10.7908$.

(d) *80% CI of* μ *=(9.2092,10.7908).*

The same situation of finding confidence intervals when 25 elements are drawn under three different levels of significance, α, is presented by the last three examples. The only difference between them is the value of α. A similar comparison was performed earlier in Example 4 but in the case when σ^2 was known. Between these two sets of examples, a combination of six confidence intervals as shown in the following table offers an opportunity to observe the consequences of knowing and not knowing the true value of σ^2 on the length of a confidence interval and the effects of imposing more or less stringent criteria in the specification of α for bracketing the population mean.

$100(1-\alpha)\%$	σ^2 Known	σ^2 Unknown
95%	(8.825,11.176)	(8.7616,11.2384)
90%	(9.016,10.984)	(8.9734,11.0266)
80%	(9.232,10.768)	(9.2092,10.7908)

The two confidence intervals corresponding to $\alpha = .05$ are shown superimposed on each other in the following diagram.



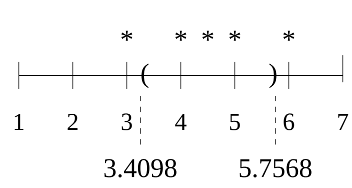
The confidence interval which is derived from the experiment in which σ^2 is known relies on the z quantile, and it is shorter than the confidence interval which is associated with an unknown σ^2. In the latter case, the construction of the confidence interval had to extract additional information from the pool of data in order to estimate σ^2 at the expense of providing a better estimate of μ. As a consequence, the estimate of μ, being less certain, requires a wider confidence interval to bracket it.

Example 7. *Let $\mathcal{S} = \{5\ 4\ 3\ 6\ 5\ 4.5\}$. Find 95% CI for μ when it is given that $\bar{x} = 4.5833$ $s^2 = 1.0417$.*

1. $\alpha = .05 \rightarrow \frac{\alpha}{2} = .025$ *and n-1=5.*
2. $t_{n-1,\frac{\alpha}{2}} = t_{5,.025} = 2.571.$
3. *lower limit,* $a = \bar{x} - \frac{s}{\sqrt{n}} t_{n-1,\frac{\alpha}{2}} = 4.5833 - \frac{\sqrt{1.0417}}{\sqrt{6}} 2.571 = 3.5120.$

   upper limit, $b = \bar{x} + \frac{s}{\sqrt{n}} t_{n-1,\frac{\alpha}{2}} = 4.5833 + \frac{\sqrt{1.0417}}{\sqrt{6}} 2.571 = 5.6546.$
4. *95% CI about μ is (3.5120,5.6546).*

Drawing a picture of the data is always the first thing to do when analyzing data. Although it is not normally done, the 95% CI is also drawn in the picture to illustrate its relation with the data. Whether it contains the population mean cannot be answered. At best, we can surmise that this might be one of the 95% confidence intervals that does cover the population mean. Besides having been given only the raw data, if a description of the context of the problem had also been given, then it would have helped in interpreting the meaning of a confidence interval. Suppose it was mentioned that this last set of data contained the geographic location of undeniable metallic objects like Spanish Doubloons of a sunken treasure in the ocean for which you are searching. The readings of magnetic sensors help to reduce the search from the entire ocean to a feasible area to look. The best place to dive into the water in search of treasure is in the middle of the confidence interval. The confidence interval marks our best guess of the location of the population mean.



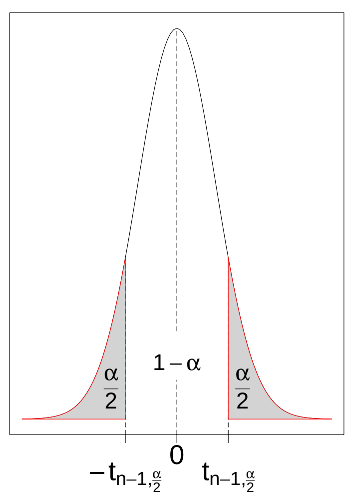
The following table summarizes the question of when to use $z_{\frac{\alpha}{2}}$ or to use $t_{n-1,\frac{\alpha}{2}}$ in constructing a confidence interval.

Obviously, $\mu - \frac{\sigma}{\sqrt{n}} z_{\frac{\alpha}{2}}$ is not a random variable, but $\bar{x} - \frac{s}{\sqrt{n}} t_{n-1,\frac{\alpha}{2}}$ is a random variable. It is the lower limit of a confidence interval. The upper limit is also a random variable. The

	When σ^2 is Known	When σ^2 is Unknown
Lower Limit	a $= \bar{x} - \frac{\sigma}{\sqrt{n}} z_{\frac{\alpha}{2}}$	a $= \bar{x} - \frac{s}{\sqrt{n}} t_{n-1,\frac{\alpha}{2}}$
Upper Limit	b $= \bar{x} + \frac{\sigma}{\sqrt{n}} z_{\frac{\alpha}{2}}$	b $= \bar{x} + \frac{s}{\sqrt{n}} t_{n-1,\frac{\alpha}{2}}$

end points of a confidence interval are random variables. They move around depending on the luck of the draw when sampling. Since $\bar{x} - \frac{s}{\sqrt{n}} t_{n-1,\frac{\alpha}{2}}$ and $\bar{x} + \frac{s}{\sqrt{n}} t_{n-1,\frac{\alpha}{2}}$ are random variables, they have probabilities distributions. Consider then the problem of finding $P(\bar{x} - \frac{s}{\sqrt{n}} t_{n-1,\frac{\alpha}{2}} \leq \mu \leq \bar{x} + \frac{s}{\sqrt{n}} t_{n-1,\frac{\alpha}{2}})$. The usual strategy of finding the z-score will lead along the following series of steps to the answer.

$$\begin{aligned}
P(\bar{x} - \frac{s}{\sqrt{n}} t_{n-1,\frac{\alpha}{2}} \leq \mu \leq \bar{x} + \frac{s}{\sqrt{n}} t_{n-1,\frac{\alpha}{2}}) &= \\
P(\bar{x} - \frac{s}{\sqrt{n}} t_{n-1,\frac{\alpha}{2}} - \bar{x} \leq \mu - \bar{x} \leq \bar{x} + \frac{s}{\sqrt{n}} t_{n-1,\frac{\alpha}{2}} - \bar{x}) &= \\
P(-\frac{s}{\sqrt{n}} t_{n-1,\frac{\alpha}{2}} \leq \mu - \bar{x} \leq \frac{s}{\sqrt{n}} t_{n-1,\frac{\alpha}{2}}) &= \\
P(-t_{n-1,\frac{\alpha}{2}} \leq \frac{\mu - \bar{x}}{\frac{s}{\sqrt{n}}} \leq t_{n-1,\frac{\alpha}{2}}) &= \\
P(-t_{n-1,\frac{\alpha}{2}} \leq t_{n-1} \leq t_{n-1,\frac{\alpha}{2}}) &= 1 - \alpha
\end{aligned}$$



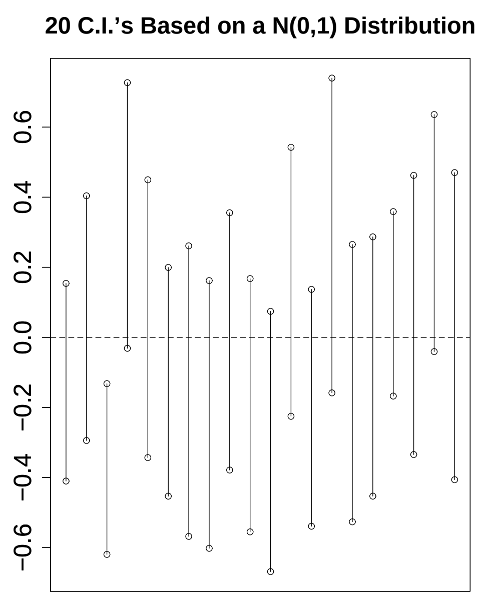
The probability that a confidence interval covers the population mean is $1 - \alpha$. This statement can be written in two informative ways: the probability that μ is in the confidence interval

$$P(\mu \in (\bar{x} - \frac{s}{\sqrt{n}} t_{n-1,\frac{\alpha}{2}},\ \bar{x} + \frac{s}{\sqrt{n}} t_{n-1,\frac{\alpha}{2}})) \quad = \quad 1 - \alpha$$

and the probability that the confidence interval covers μ

$$P((\bar{x} - \frac{s}{\sqrt{n}} t_{n-1,\frac{\alpha}{2}},\ \bar{x} + \frac{s}{\sqrt{n}} t_{n-1,\frac{\alpha}{2}}) \ni \mu) \quad = \quad 1 - \alpha$$

At this point, we have reached a philosophical dilemma. The endpoints, $(\bar{x} - \frac{s}{\sqrt{n}} t_{n-1,\frac{\alpha}{2}},\ \bar{x} + \frac{s}{\sqrt{n}} t_{n-1,\frac{\alpha}{2}})$, of a confidence interval are random variables. When values of experimental data are substituted into the formula as was done in Example 7, a realized confidence interval, (3.5120, 5.6546) is produced. Its endpoints are constants; consequently, it is not a confidence interval and $P((3.5120, 5.6546) \ni \mu)$ makes no sense because it is indeterminable. It is obvious that $P((3.5120, 5.6546) \ni 0) = 0$ and $P((3.5120, 5.6546) \ni 4) = 1$, but with μ unknown, it is impossible to tell what $P((3.5120, 5.6546) \ni \mu)$ is. When values of experimental data are substituted into the formula for a confidence interval, we immediately leave the world of probability; therefore, it is wrong to frame a realized confidence interval in terms of probability and to say, for instance, that (3.5120, 5.6546) covers the mean of the population with 95 percent probability or to say that the mean of the population is contained in (3.5120, 5.6546) with a probability of 95 percent. On the other hand, it is correct to say that $(\bar{x} - \frac{s}{\sqrt{n}} t_{n-1,\frac{\alpha}{2}},\ \bar{x} + \frac{s}{\sqrt{n}} t_{n-1,\frac{\alpha}{2}})$ covers the population mean with $100(1-\alpha)\%$ probability because $\bar{x} \pm \frac{s}{\sqrt{n}} t_{n-1,\frac{\alpha}{2}}$ are random variables as opposed to 3.5120 and 5.6546 which are constants. The answer to this philosophical dilemma relies on the recognition that (3.5120, 5.6546) is a realized value of a confidence interval. It may or may not cover the mean of the population, and we are left wondering which situation is correct.



If 100 independent and identical experiments are conducted from which 100 realized confidence intervals are constructed, then the theory of confidence intervals says that, on the average, 95 of those intervals will contain the population mean. It is impossible to identify which one of them does contain the mean and which one does not contain the mean.

To illustrate this important concept, a series of 20 computer simulated experiments were conducted in which the population mean is known to be zero. In each experiment, 30 random numbers were generated by means of a computer from a N(0,1) distribution. The 20 resulting confidence intervals about the mean are stacked side by side each other as shown in the drawing above. If the line marking the location of the population mean was not present as is the case in an actual experiment, it would be impossible to tell which confidence intervals contain $\mu = 0$. In the contrived experiment presented here, the theory of confidences intervals is confirmed for we see that 19/20=95% of the realized confidence intervals do contain $\mu = 0$.

Statisticians almost never say *realized confidence interval* except in philosophical discussions. It should be kept in mind that confidence interval and a realized confidence interval are two different things. For the pragmatist, the philosophical distinction between them is overlooked. In his view, that interval which is produced from experimental data is his best bet for the location of the population mean.

Another way to look at 95% confidence intervals is to imagine a stack of 100 confidence intervals on top of each other like a stack of pancakes. The resulting histogram will look like a Normal distribution with mean $\bar{x}$ and standard deviation $s = \frac{(b-a)\sqrt{n}}{2t_{n-1,\frac{\alpha}{2}}}$.

If the Uniform distribution is the simplest one among those for continuous random variables and if the Normal distribution is the nicest of all distributions, then the Student's t distribution is one of the the simplest of the complex distributions. The formula of its probability density function with n degrees of freedom is:

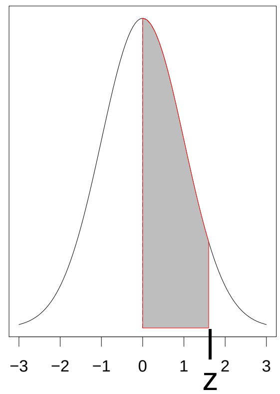
$$f(x) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi n}\Gamma(\frac{n}{2})(1+\frac{x^2}{n})^{\frac{n+1}{2}}}$$

where $\Gamma(z) = \int\limits_0^\infty t^{z-1}e^{-t}dt$.

The gamma function, $\Gamma(z)$, is a celebrated example of Leonhard Euler's ingenuity. He generalized the factorial function, $N!$, to allow in its domain not only whole numbers but all numbers except negative integers and zero. It is very easy to evaluate $3! = 3 \cdot 2 \cdot 1$ and $4! = 4 \cdot 3 \cdot 2 \cdot 1$ and it seems as if there should be a factorial of a value between them like $3\frac{1}{2}!$. In fact, there is such a value. By means of the gamma function which Euler discovered, $N! = \Gamma(N+1)$ so that it is possible to evaluate, for instance $(-\frac{1}{2})! = \Gamma(\frac{1}{2}) = \sqrt{\pi}$, another remarkable equivalence involving π. Since 3!=6 and 4!=24, it would seem that $6 \leq 3\frac{1}{2}! \leq 24$. Indeed, by using the definition of factorial, we can write $3\frac{1}{2}! = \frac{7}{2}! = \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}(-\frac{1}{2})! = \frac{105}{16}\sqrt{\pi} = 11.63173\ldots$.

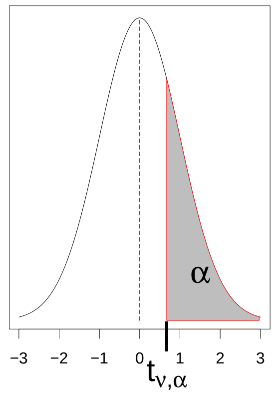
In conclusion of this discussion of the Student's t distribution, if $T \sim t_n$, then E[T]=0 and $var(T) = \frac{n}{n-2}$. Because E[T]=0, the t distribution technically should be called the central t distribution. There is a large family of t distributions for which $E[T] \neq 0$; they are called non-central t distributions, and they are used in advanced statistics.

Student's t T_n
$E[T_n] = 0$
$var(T_n) = \frac{n}{n-2}$



Cumulative Probabilities for a N(0,1) Distribution: $\Phi(z) - .5$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.00000	0.00399	0.00798	0.01197	0.01595	0.01994	0.02392	0.0279	0.03188	0.03586
0.1	0.03983	0.04380	0.04776	0.05172	0.05567	0.05962	0.06356	0.06749	0.07142	0.07535
0.2	0.07926	0.08317	0.08706	0.09095	0.09483	0.09871	0.10257	0.10642	0.11026	0.11409
0.3	0.11791	0.12172	0.12552	0.12930	0.13307	0.13683	0.14058	0.14431	0.14803	0.15173
0.4	0.15542	0.15910	0.16276	0.16640	0.17003	0.17364	0.17724	0.18082	0.18439	0.18793
0.5	0.19146	0.19497	0.19847	0.20194	0.20540	0.20884	0.21226	0.21566	0.21904	0.22240
0.6	0.22575	0.22907	0.23237	0.23565	0.23891	0.24215	0.24537	0.24857	0.25175	0.25490
0.7	0.25804	0.26115	0.26424	0.26730	0.27035	0.27337	0.27637	0.27935	0.28230	0.28524
0.8	0.28814	0.29103	0.29389	0.29673	0.29955	0.30234	0.30511	0.30785	0.31057	0.31327
0.9	0.31594	0.31859	0.32121	0.32381	0.32639	0.32894	0.33147	0.33398	0.33646	0.33891
1.0	0.34134	0.34375	0.34614	0.34849	0.35083	0.35314	0.35543	0.35769	0.35993	0.36214
1.1	0.36433	0.36650	0.36864	0.37076	0.37286	0.37493	0.37698	0.37900	0.38100	0.38298
1.2	0.38493	0.38686	0.38877	0.39065	0.39251	0.39435	0.39617	0.39796	0.39973	0.40147
1.3	0.40320	0.40490	0.40658	0.40824	0.40988	0.41149	0.41309	0.41466	0.41621	0.41774
1.4	0.41924	0.42073	0.42220	0.42364	0.42507	0.42647	0.42785	0.42922	0.43056	0.43189
1.5	0.43319	0.43448	0.43574	0.43699	0.43822	0.43943	0.44062	0.44179	0.44295	0.44408
1.6	0.44520	0.44630	0.44738	0.44845	0.44950	0.45053	0.45154	0.45254	0.45352	0.45449
1.7	0.45543	0.45637	0.45728	0.45818	0.45907	0.45994	0.46080	0.46164	0.46246	0.46327
1.8	0.46407	0.46485	0.46562	0.46638	0.46712	0.46784	0.46856	0.46926	0.46995	0.47062
1.9	0.47128	0.47193	0.47257	0.47320	0.47381	0.47441	0.47500	0.47558	0.47615	0.47670
2.0	0.47725	0.47778	0.47831	0.47882	0.47932	0.47982	0.48030	0.48077	0.48124	0.48169
2.1	0.48214	0.48257	0.48300	0.48341	0.48382	0.48422	0.48461	0.48500	0.48537	0.48574
2.2	0.48610	0.48645	0.48679	0.48713	0.48745	0.48778	0.48809	0.48840	0.48870	0.48899
2.3	0.48928	0.48956	0.48983	0.49010	0.49036	0.49061	0.49086	0.49111	0.49134	0.49158
2.4	0.49180	0.49202	0.49224	0.49245	0.49266	0.49286	0.49305	0.49324	0.49343	0.49361
2.5	0.49379	0.49396	0.49413	0.49430	0.49446	0.49461	0.49477	0.49492	0.49506	0.49520
2.6	0.49534	0.49547	0.49560	0.49573	0.49585	0.49598	0.49609	0.49621	0.49632	0.49643
2.7	0.49653	0.49664	0.49674	0.49683	0.49693	0.49702	0.49711	0.49720	0.49728	0.49736
2.8	0.49744	0.49752	0.49760	0.49767	0.49774	0.49781	0.49788	0.49795	0.49801	0.49807
2.9	0.49813	0.49819	0.49825	0.49831	0.49836	0.49841	0.49846	0.49851	0.49856	0.49861
3.0	0.49865	0.49869	0.49874	0.49878	0.49882	0.49886	0.49889	0.49893	0.49896	0.49900



Quantiles for a Student's t Distribution

ν	$t_{\nu,.20}$	$t_{\nu,.15}$	$t_{\nu,.10}$	$t_{\nu,.05}$	$t_{\nu,.025}$	$t_{\nu,.01}$	$t_{\nu,005}$
1	1.37638	1.96261	3.07768	6.31375	12.7062	31.82052	63.65674
2	1.06066	1.38621	1.88562	2.91999	4.30265	6.964560	9.92484
3	0.97847	1.24978	1.63775	2.35338	3.18245	4.54070	5.84091
4	0.94096	1.18957	1.53321	2.13185	2.77645	3.74695	4.60410
5	0.91954	1.15577	1.47588	2.01505	2.57058	3.36493	4.03216
6	0.90570	1.13416	1.43976	1.94318	2.44691	3.14267	3.70743
7	0.89603	1.11916	1.41492	1.89458	2.36462	2.99795	3.49948
8	0.88889	1.10815	1.39682	1.85955	2.30600	2.89646	3.35539
9	0.88340	1.09972	1.38303	1.83311	2.26216	2.82144	3.24984
10	0.87906	1.09306	1.37218	1.81246	2.22814	2.76377	3.16927
11	0.87553	1.08767	1.36343	1.79588	2.20099	2.71808	3.10581
12	0.87261	1.08321	1.35622	1.78229	2.17881	2.68100	3.05454
13	0.87015	1.07947	1.35017	1.77093	2.16037	2.65031	3.01228
14	0.86805	1.07628	1.34503	1.76131	2.14479	2.62449	2.97684
15	0.86624	1.07353	1.34061	1.75305	2.13145	2.60248	2.94671
16	0.86467	1.07114	1.33676	1.74588	2.11991	2.58349	2.92078
17	0.86328	1.06903	1.33338	1.73961	2.10982	2.56693	2.89823
18	0.86205	1.06717	1.33039	1.73406	2.10092	2.55238	2.87844
19	0.86095	1.06551	1.32773	1.72913	2.09302	2.53948	2.86093
20	0.85996	1.06402	1.32534	1.72472	2.08596	2.52798	2.84534
21	0.85907	1.06267	1.32319	1.72074	2.07961	2.51765	2.83136
22	0.85827	1.06145	1.32124	1.71714	2.07387	2.50832	2.81876
23	0.85753	1.06034	1.31946	1.71387	2.06866	2.49987	2.80734
24	0.85686	1.05932	1.31784	1.71088	2.06390	2.49216	2.79694
25	0.85624	1.05838	1.31635	1.70814	2.05954	2.48511	2.78744
26	0.85567	1.05752	1.31497	1.70562	2.05553	2.47863	2.77871
27	0.85514	1.05673	1.31370	1.70329	2.05183	2.47266	2.77068
28	0.85465	1.05599	1.31253	1.70113	2.04841	2.46714	2.76326
29	0.85419	1.05530	1.31143	1.69913	2.04523	2.46202	2.75639
30	0.85377	1.05466	1.31042	1.69726	2.04227	2.45726	2.75000
40	0.85070	1.05005	1.30308	1.68385	2.02108	2.42326	2.70446
50	0.84887	1.04729	1.29871	1.67591	2.00856	2.40327	2.67779
75	0.84644	1.04365	1.29294	1.66543	1.99210	2.37710	2.64298
100	0.84523	1.04184	1.29007	1.66023	1.98397	2.36422	2.62589
150	0.84402	1.04003	1.28722	1.65508	1.97591	2.35146	2.60900
∞	0.84162	1.03643	1.28155	1.64485	1.95996	2.32635	2.57583